Impact of positivity and complete positivity on accessibility of Markovian dynamics

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Abstract

We consider a two-dimensional quantum control system evolving under an entropy-increasing irreversible dynamics in the semigroup form. Considering a phenomenological approach to the dynamics, we show that the accessibility property of the system depends on whether its evolution is assumed to be positive or completely positive. In particular, we characterize the family of maps having different accessibility and show the impact of that property on observable quantities by means of a simple physical model.

1 Introduction

Irreversible dynamics appear in many fields of atomic and nuclear physics [1] and in quantum chemistry. They have great relevance in quantum optics [2, 3], in statistical mechanics [4] and in the description of continuously measured systems [5]. They describe the time evolution of a physical system in interaction with a second system (usually the external environment). Under rather mild assumptions (as for example a weak interaction between system and environment) these dynamics are Markovian, that is they consist of semigroups of maps whose generator for an n-level system has the standard form

$$\dot{\rho} = -i[H, \rho] + \sum_{k,l=1}^{n^2-1} c_{kl} \left(F_k \rho F_l^\dagger - \frac{1}{2} \left\{ F_l^\dagger F_k, \rho \right\} \right), \tag{1}$$

where ρ is the density matrix associated to the system (i.e. a $n \times n$ positive matrix with unit trace), H is an Hermitian operator and the set $\{F_k, k = 1, \ldots, n^2-1\}$ satisfies $\mathrm{Tr} F_k^\dagger F_l = \delta_{kl}$, $k, l = 0, \ldots, n^2-1$ and $F_0 = I/\sqrt{n}$. The $(n^2-1) \times (n^2-1)$ matrix C with entries c_{kl} is Hermitian; for entropy increasing time evolutions it is real and symmetric [6, 7].

From (1) we get $\rho_t = \gamma_t[\rho_{in}]$ where $\{\gamma_t, t \geqslant 0\}$ is a one-parameter semigroup of maps describing the time evolution of the system, and ρ_{in}, ρ_t are the initial and final (at time t) states respectively. These dynamics must fulfill some requirements necessary for a consistent interpretation of the mathematical formalism. In particular, at any time t they must preserve the positivity and the trace of the density matrix they act over. The evolutions generated by (1) are automatically trace-preserving, however some constraints on the coefficients c_{kl} have to be imposed in order to preserve the positivity of ρ_t.

Definition 1.1 The map γ_t is said to be *positivity-preserving* or simply *positive* if and only if $\gamma_t[\rho] \geqslant 0 \; \forall \, \rho \geqslant 0$.

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Definition 1.2 The map γ_t is said to be *completely positive* if and only if $\gamma_t \otimes I_n$ is positive $\forall\, n \in \mathbb{N}$, with I_n the n-dimensional identity.

By definition, complete positivity implies positivity. It is usually believed (or assumed) that the evolutions generated by (1) should be completely positive rather than simply positivity-preserving [6, 7]. In fact, whereas positivity-preserving maps guarantee that ρ_t remains positive at any time, the stronger property of complete positivity is necessary to preserve the positivity of states initially entangled with the environment (or part of it) [6, 7, 8, 9].

Remark 1.3 The dynamics generated by (1) is completely positive if and only if $C \geqslant 0$ [6, 7].

It is worth to consider that equation (1) is the Markovian approximation of an exact, irreversible dynamics. This dynamics, for an open system, is given by $\rho_t = \mathrm{Tr}_E(\mathcal{U}_t \rho_0)$, where ρ_0 is the initial state of the composite system (open system + environment), $\mathcal{U}_t$ its unitary time evolution and Tr_E the partial trace over the environment degrees of freedom. For an uncorrelated initial state, $\rho_0 = \rho_{in} \otimes \rho_E$ (where ρ_E is a reference state in the environment) this dynamics is completely positive. Indeed it is the composition of three completely positive maps, the expansion $\rho_{in} \to \rho_{in} \otimes \rho_E$, the unitary evolution $\mathcal{U}_t$ and the partial trace Tr_E. From this point of view, the preservation of complete positivity is a rather natural assumption once a Markovian approximation of the dynamics is assumed[1].

Although Markovian approximations leading to completely positive time evolutions are always possible [18] and, following the previous discussion, desirable, they are not universally adopted: sometimes the simple positivity is asked for [12, 13, 14, 15, 16].

There are two approaches leading to the generator (1): an axiomatic and a constructive one [17]. In the first case both the form (1) and the complete positivity of the resulting time evolution are assumed (this is the usual approach in the quantum theories of information and computation). In the latter case, (1) is derived as the Markovian approximation of a generalized master equation, and complete positivity or positivity are imposed as a second step. This procedure leads to phenomenological coefficients c_{kl} that are fixed but in general unknown, since they embody the microscopic details of the interaction between the open system and its external environment, usually not accessible. In general, these coefficients are not all independent and the assumptions of positivity and complete positivity of the dynamics add further constraints.

Complete positivity is physically motivated introducing the correlations of the considered system with an arbitrary external system. Moreover it implies a hierarchy on the relaxing times of the elements of ρ_t (the diagonal with respect to the off-diagonal ones) absent if the weaker property of positivity is asked for [6]. For these reasons in the previously cited works simply positivity-preserving maps are preferred to completely positive ones: to their authors, complete positivity appears as an artificial mathematical request affecting too strongly the physical behavior of the system.

In the following we will assume the constructive point of view for a Markovian dynamics and address the problem of the choice between positivity and complete positivity from a control theoretical point of view. We will show that for some families of dynamics this choice leads to different accessibility properties and, in turn, to detectable differences in some observable quantity of the system.

[1] This argument does not hold for a non-separable initial state ρ_0 (for more details, see ref.[10], [11])

We limit our attention to 2-level systems evolving under entropy increasing irreversible evolutions. This case describes many interesting physical apparatuses, moreover it presents a simple characterization of positive maps. We choose as basis operators the Pauli matrices, $F_k = \sigma_k$, $k = 1, 2, 3$ and $\sigma_0 = I$, and write a coherent vector representation of (1):

$$\dot{\vec{\rho}} = \mathcal{L}\vec{\rho} = -(\mathcal{H} + \mathcal{D})\vec{\rho}, \tag{2}$$

where we defined the vector $\vec{\rho} = (\rho_1, \rho_2, \rho_3)$ with $\rho_k = \mathrm{Tr}(\rho\sigma_k)/2$, $k = 1, 2, 3$. Since the time evolution is trace-preserving, $\rho_0 = \mathrm{Tr}(\rho\sigma_0)/2 = 1/2$ is a constant. Positivity of ρ translate to $\| \vec{\rho} \|^2 = \langle \vec{\rho}, \vec{\rho} \rangle \leqslant 1/4$, defining the Bloch sphere $\mathbb{B}_0(1/2)$. The 3×3 real matrices $\mathcal{H}$ and $\mathcal{D}$ are skew-symmetric, respectively symmetric and represent the Hamiltonian (or coherent) contribution and the dissipative one.

Remark 1.4 The entropy-increasing dynamics generated by (1) for a two-level system is positivity-preserving if and only if $\mathcal{D} \geqslant 0$ [6].

So, in this framework both positivity and complete positivity of the dynamics are fully characterized by the positivity of 3×3 matrices.

We further assume that the dynamics can be externally modified by m control functions $u_1, \ldots, u_m$, affecting the Hamiltonian contribution, that is $H = H_0 + u_1 H_1 + \ldots + u_m H_m$. In the coherence vector formalism,

$$\dot{\vec{\rho}} = -(\mathcal{H}_0 + \sum_{i=1}^{m} u_i \mathcal{H}_i + \mathcal{D})\vec{\rho}, \quad \vec{\rho}(0) = \vec{\rho}_i. \tag{3}$$

Integrating (3) we get a multi-parameter semigroup of time evolutions. Our aim is to characterize the cases where the controllability and accessibility properties can be different under the requests of positivity or complete positivity.

We say that $\vec{\rho}'$ can be reached from $\vec{\rho}$ at time t if there exist some controls u_i, $i = 1, \ldots, m$ such that the time evolution generated by (3) steers $\vec{\rho}$ to $\vec{\rho}'$ at time t. The set of all $\vec{\rho}'$ which are attainable from $\vec{\rho}$ at time t is denoted by $\mathcal{R}(\vec{\rho}, t)$. The following definitions and properties are standard in Control Theory [19].

Definition 1.5 The *reachable set from $\vec{\rho}_i$ at time T* for the system (3) is given by

$$\mathcal{R}(\vec{\rho}_i, T) = \bigcup_{0 \leqslant t \leqslant T} \mathcal{R}(\vec{\rho}_i, t).$$

Definition 1.6 The *reachable set from $\vec{\rho}_i$* for the system (3) is given by

$$\mathcal{R}(\vec{\rho}_i) = \bigcup_{t \geqslant 0} \mathcal{R}(\vec{\rho}_i, t).$$

These sets depend on the particular choice of the initial state $\vec{\rho}_i$.

Definition 1.7 System (3) is *accessible* if and only if $\mathcal{R}(\vec{\rho}_i, T)$ contains nonempty open sets of $\mathbb{B}_0(1/2)$ $\forall T > 0$.

From a physical point of view, accessibility means that the system can be driven in every direction in the state space. If a system is not accessible, there are forbidden directions in the state space for the time evolution of the initial state. This affects the reachable sets in the two cases. However, the evaluation of the reachable sets is usually more difficult than the study of the accessibility property. In fact, accessibility can be expressed by a simple property, the so-called Lie algebra rank condition (LARC).

Remark 1.8 System (3) is accessible if and only if $Lie(\mathcal{H}_0 + \mathcal{D}, \mathcal{H}_1, \dots, \mathcal{H}_m))$ is transitive on $\mathbb{B}_0(1/2)$, that is $Lie(\mathcal{H}_0 + \mathcal{D}, \mathcal{H}_1, \dots, \mathcal{H}_m) = \mathfrak{gl}(3, \mathbb{R})$ or $\mathfrak{sl}(3, \mathbb{R})$.

Definition 1.9 System (3) is *controllable* if and only if $\forall\, (\rho_i, \rho_f) \in \mathbb{B}_0(1/2) \times \mathbb{B}_0(1/2)$ there is a set of controls $u_1, \dots, u_m$ such that $\vec{\rho}(0) = \vec{\rho}_i$ and $\vec{\rho}(t) = \vec{\rho}_f$ for some $t \geqslant 0$.

This means that we can steer every initial state to an arbitrary final state using a suitable succession of controls. Otherwise said, $\mathcal{R}(\vec{\rho}_i) = \mathbb{B}_0(1/2)$ for every initial state $\vec{\rho}_i$: we can arbitrarily move into the state space using the control functions.

The controllability properties of quantum mechanical systems have been characterized for both closed systems (see [20] and references therein) and open ones. In particular, in the latter case both accessibility and controllability properties have been fully investigated in [21] under the assumption of complete positivity. In particular, the system is not controllable since $d/dt \parallel \vec{\rho} \parallel^2 = 2(\dot{\vec{\rho}}, \vec{\rho}) = -(\mathcal{D}\vec{\rho}, \vec{\rho}) \leqslant 0$. This condition holds true even if we assume positivity instead of complete positivity, therefore the results concerning controllability are the same under the constraints of positivity and complete positivity. However, the system could be accessible or not in the two cases.

As a first step, in Section 2 we discuss how the requests of positivity and complete positivity affect the space of parameters associated to a dissipative dynamics. In particular, Theorem 1 characterizes those dynamics for which the constraints of positivity and complete positivity lead to different - but not trivial - generators, a necessary condition for different accessibility properties. In Section 3 we consider this class of maps and, using the LARC, we show that Theorem 1 is not a sufficient condition for a different accessibility. After exploring a concrete example of control system belonging to this family, we discuss our results.

2 Necessary conditions for different accessibility properties

Given a generator in the form (1), we assume that the entries of C are either independent or linearly dependent unknown phenomenological parameters. Neglecting upper bounds on their values[2], there is a one-to-one correspondence between the matrices C and the linear spaces $\mathcal{V} \subseteq \mathbb{R}^6$. The number of independent entries of C is given by $n = dim\, \mathcal{V}$.

The requests of positivity and complete positivity define two convex cones in the linear space of 3×3 real symmetric matrices, $\mathscr{C}_p$ (given by $\mathcal{D} \geqslant 0$) and $\mathscr{C}_{cp}$ ($C \geqslant 0$). Since complete positivity is a stronger property than positivity, $\mathscr{C}_{cp} \subset \mathscr{C}_p$. We consider $\mathscr{S}_p = \mathcal{V} \cap \mathscr{C}_p$ and $\mathscr{S}_{cp} = \mathcal{V} \cap \mathscr{C}_{cp}$, the sets of matrices C restricted by the conditions of positivity or complete positivity, and $\mathscr{S}_{cp} \subseteq \mathscr{S}_p$. An arbitrary $\mathcal{V} \in \mathbb{R}^6$ will fit one of the following cases (0 is the null matrix):

1. $\mathscr{S}_p = \{0\}$ and $\mathscr{S}_{cp} = \{0\}$;

[2] This assumption won't affect our results

2. $\mathscr{S}_p \neq \{0\}$ and $\mathscr{S}_{cp} = \{0\}$,
   (a) $\mathscr{S}_p \subset \partial\mathscr{C}_p$,
   (b) $\mathscr{S}_p \nsubseteq \partial\mathscr{C}_p$;
3. $\mathscr{S}_p \neq \{0\}$ and $\mathscr{S}_{cp} \neq \{0\}$,
   (a) $\mathscr{S}_p \subset \partial\mathscr{C}_p$ and $\mathscr{S}_{cp} \subset \partial\mathscr{C}_{cp}$,
   (b) $\mathscr{S}_p \nsubseteq \partial\mathscr{C}_p$ and $\mathscr{S}_{cp} \subset \partial\mathscr{C}_{cp}$,
   (c) $\mathscr{S}_p \nsubseteq \partial\mathscr{C}_p$ and $\mathscr{S}_{cp} \nsubseteq \partial\mathscr{C}_{cp}$.

In case 1 any dynamics does not preserve the positivity of the states over which it acts, in case 2 there are not completely positive dynamics, while in case 3 both positive and completely positive time evolutions can be obtained choosing the entries of C.

The number of independent entries of C after the requests of positivity or complete positivity is equal to the dimension of the smaller linear space containing the sets $\mathscr{S}_p$, $\mathscr{S}_{cp}$: $\mathcal{V}_p = span\{\mathscr{S}_p\}$ and $\mathcal{V}_{cp} = span\{\mathscr{S}_{cp}\}$. We define $n_p = dim\,\mathcal{V}_p$ and $n_{cp} = dim\,\mathcal{V}_{cp}$. In general $\mathcal{V}_{cp} \subseteq \mathcal{V}_p$, then $n_{cp} \leqslant n_p \leqslant n$. Different accessibility properties under the requests of positivity and complete positivity are possible only if $n_{cp} < n_p$. Otherwise, although the constraints between the dissipative parameters are different in the two cases, we get the same Lie algebra and then, via the LARC, accessibility is independent on the constraint imposed.

In order to compute n_p and n_{cp} in every case, we derive some relevant properties of the tangent space at the boundary of the cone of real, positive and symmetric 3×3 matrices, $\partial\mathscr{C}$. The inner product in $\mathbb{R}^6$ will be denoted by $\langle \cdot\,,\,\cdot \rangle$.

Lemma 2.1 The 5-dimensional linear space $\mathcal{T}$ is the tangent space in some $P \in \partial\mathscr{C}$ if and only if $\exists\, \vec{v} \in \mathbb{R}^3$, $\vec{v} \neq \vec{0}$, such that $\langle \vec{v}, T\vec{v} \rangle = 0\ \forall\, T \in \mathcal{T}$.
Proof: The boundary $\partial\mathscr{C}$ is given by the set of positive matrices with at least one vanishing eigenvalue. In fact, since P is real and symmetric, it can be put in diagonal form by means of an orthogonal transformation, that is $P = \mathcal{O}\tilde{P}\mathcal{O}^T$, $\mathcal{O} \in O(3)$ and $\tilde{P}$ a diagonal matrix with $(\tilde{P})_{ii} = 0$ for some $i \in \{1,2,3\}$. Except for the null matrix, $\partial\mathscr{C}_p$ is a smooth 5-dimensional manifold and its tangent spaces are 5-dimensional linear spaces. In particular, $\mathcal{T} = \mathcal{O}\tilde{\mathcal{T}}\mathcal{O}^T$ where $\tilde{\mathcal{T}}$ is the tangent space in $\tilde{P}$. Since $(\tilde{P})_{ii} = 0$ for some $i \in \{1,2,3\}$, it follows $(\tilde{T})_{ii} = 0$ $\forall\, \tilde{T} \in \tilde{\mathcal{T}}$. Consequently $\langle \vec{e}_i, \tilde{T}\vec{e}_i \rangle = 0\ \forall\, \tilde{T} \in \tilde{\mathcal{T}}$, where $\vec{e}_i$ is defined by $(\vec{e}_i)_j = \delta_{ij}$, and finally $\langle \vec{v}, T\vec{v} \rangle = 0\ \forall\, T \in \mathcal{T}$, with $\vec{v} = \mathcal{O}\vec{e}_i$. $\square$

Lemma 2.2 If $\mathcal{T}$ is a tangent space of the manifold $\partial\mathscr{C}$, then $span\{\partial\mathscr{C}_p \cap \mathcal{T}\}$ is a 3-dimensional linear subspace of $\mathcal{T}$.
Proof: The linear spaces $span\{\partial\mathscr{C} \cap \mathcal{T}\}$ and $span\{\partial\mathscr{C} \cap \tilde{\mathcal{T}}\}$ are isomorphic. Given $\tilde{T} \in \partial\mathscr{C} \cap \tilde{\mathcal{T}}$, by Lemma 2.1 and since $\tilde{T} \geqslant 0$ the i-th row and column of $\tilde{T}$ must vanish. Then $span\{\partial\mathscr{C} \cap \tilde{\mathcal{T}}\}$ is 3-dimensional and the thesis follows. $\square$

Given an arbitrary C, the matrix $\mathcal{D}$ can be expressed in terms of its entries:

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}, \quad \mathcal{D} = 2\begin{pmatrix} c_{22}+c_{33} & -c_{12} & -c_{13} \\ -c_{12} & c_{11}+c_{33} & -c_{23} \\ -c_{13} & -c_{23} & c_{11}+c_{22} \end{pmatrix} \tag{4}$$

that is $\mathcal{D} = 2(I\,\mathrm{Tr}\,C - C)$. Since there is an isomorphism between unitary transformations of the Pauli matrices, $\tilde{\sigma}_i = \mathcal{U}\sigma_i\mathcal{U}^\dagger$ with $\mathcal{U} \in U(2)$, and orthogonal transformations of C (and $\mathcal{D}$), $\tilde{C} = \mathcal{O}^T C \mathcal{O}$ (and $\tilde{\mathcal{D}} = \mathcal{O}^T \mathcal{D} \mathcal{O}$) with $\mathcal{O} \in O(3)$, the orthogonal transformations in Lemma 2.1 are related to unitary changes of basis. In particular, the relation between the entries of C and $\mathcal{D}$ is basis independent since $\tilde{\mathcal{D}} = 2(I\,\mathrm{Tr}\,\tilde{C} - \tilde{C})$. Notice also that whenever $\mathcal{V}$ is a subset of the boundary of some cone, it is a linear subspace of a tangent space to this boundary. Then, following Lemmas 2.1 end 2.2 and considering the expressions (4), n_p and n_{cp} can be evaluated in every case. They are listed in the following table:

case	n	n_p	n_{cp}
1	$\leqslant 5$	0	0
2(a)	$\leqslant 5$	$\leqslant 3$	0
2(b)	$\leqslant 5$	n	0
3(a)	$\leqslant 4$	$\leqslant 2$	1
3(b)	$\leqslant 5$	n	$\leqslant 3$
3(c)	$\leqslant 6$	n	n

The requests of positivity and complete positivity produce linear spaces $\mathcal{V}_p$ and $\mathcal{V}_{cp}$ lower dimensional than $\mathcal{V}$ in two cases. Either $\mathcal{V}$ is a linear subspace of a tangent space to the boundaries of the respective cones, or it intersects these cones only in the null matrix 0. The only non trivial cases admitting both positive and completely positive dissipative contributions with $n_p > n_{cp}$ are 3(a) and 3(b). For later reference, we present here a necessary and sufficient condition for the form of the generators of these dynamics.

Theorem 1 *Given a generic C in (1), define $K = span\{\vec{w} \in \mathbb{R}^3 | \langle \vec{w}, C\vec{w}\rangle = 0\}$. Then $n_p > n_{cp} \neq 0$ if and only if one of the two following conditions is satisfied:*

1. *$dim\,K = 1$ and $\forall\, \vec{w} \in K$, $\vec{w} \neq 0$ we have $C\vec{w} \neq 0$;*
2. *$dim\,K = 2$ and $\exists\, \vec{w}_1, \vec{w}_2 \in K$ such that $\langle \vec{w}_1, \vec{w}_2\rangle = 0$ and $\langle \vec{w}_1, C\vec{w}_2\rangle \neq 0$.*

Proof: As previously stated, $n_p > n_{cp} \neq 0$ if and only if we are in case 3(a) or 3(b).
In case 3(a), by Lemma 2.1, $\exists\, \vec{v} \in \mathbb{R}^3$, $\vec{v} \neq \vec{0}$, such that $\langle \vec{v}, \mathcal{D}\vec{v}\rangle = 0$ or, in a suitable basis, $(\tilde{\mathcal{D}})_{ii} = 0$ for some $i \in \{1,2,3\}$ (whit $\vec{v} = \mathcal{O}\vec{e}_i$ and $\tilde{\mathcal{D}} = \mathcal{O}^T\mathcal{D}\mathcal{O}$) and thus $\tilde{c}_{jj} = 0$ for $j \neq i$. Therefore $dim\,K = 2$. Moreover $n_p \neq n_{cp}$ if and only if $\tilde{c}_{jk} \neq 0$, with $j \neq k$ and $j,k \neq i$. Defining $\vec{w}_{1,2} = \mathcal{O}\vec{e}_{j,k}$ we have $\langle \vec{w}_1, \vec{w}_2\rangle = 0$ and $\langle \vec{w}_1, C\vec{w}_2\rangle \neq 0$.
In case 3(b), $\exists\, \vec{v} \in \mathbb{R}^3$, $\vec{v} \neq \vec{0}$, such that $\langle \vec{v}, C\vec{v}\rangle = 0$ or, arguing as before, $\tilde{c}_{ii} = 0$ for some $i \in \{1,2,3\}$. Then $dim\,K \geqslant 1$. Necessary and sufficient condition for $n_p \neq n_{cp}$ is $\tilde{c}_{ij} \neq 0$ for some $j \neq i$. In particular, if $dim\,K = 1$ define $\vec{w} = \mathcal{O}\vec{e}_j$; if $dim\,K = 2$ define $\vec{w}_{1,2} = \mathcal{O}\vec{e}_{j,k}$ with $j \neq k$ and $j,k \neq i$. Since complete positivity implies $\tilde{c}_{ij} = 0$ for $j \neq i$, the thesis follows. $\square$

3 Accessibility for positive and completely positive maps

In this section we study the accessibility properties of some selected evolutions fitting the conditions expressed by Theorem 1. We limit our attention to a single control u switching on/off the Hamiltonian part,

$$\mathcal{L}(u) = -(u\mathcal{H} + \mathcal{D}), \tag{5}$$

with $u = 0, 1$. The more general dissipative contribution in (2), restricted by the requests of positivity or complete positivity, up to an orthogonal transformation is given by

$$\mathcal{D}_p = 2 \begin{pmatrix} c_{22} & -c_{12} & -c_{13} \\ -c_{12} & c_{11} & -c_{23} \\ -c_{13} & -c_{23} & c_{11} + c_{22} \end{pmatrix}, \quad \mathcal{D}_{cp} = 2 \begin{pmatrix} c_{22} & -c_{12} & 0 \\ -c_{12} & c_{11} & 0 \\ 0 & 0 & c_{11} + c_{22} \end{pmatrix} \tag{6}$$

with $(c_{13}, c_{23}) \neq (0, 0)$. To simplify the computations, we assume $c_{12} = c_{13} = 0$. The Hamiltonian contribution is given by the skew-symmetric matrix

$$\mathcal{H} = 2 \begin{pmatrix} 0 & h_3 & -h_2 \\ -h_3 & 0 & h_1 \\ h_2 & -h_1 & 0 \end{pmatrix}, \tag{7}$$

where $h_k = \mathrm{Tr}(H\sigma_k)/2$, $k = 1, 2, 3$, and H in (1). Without loss of generality, we choose either $h_1 \neq 0$ and $h_2 = h_3 = 0$, or $h_1 = h_2 = 0$ and $h_3 \neq 0$.
The Lie algebras generated in the two cases are denoted by $\mathcal{A}_p$ and $\mathcal{A}_{cp}$:

$$\begin{aligned} \mathcal{A}_p &= Lie(\mathcal{D}_p, \mathcal{H} + \mathcal{D}_p) \\ \mathcal{A}_{cp} &= Lie(\mathcal{D}_{cp}, \mathcal{H} + \mathcal{D}_{cp}). \end{aligned} \tag{8}$$

A convenient orthogonal basis for the algebra of 3×3 real matrices, $\mathfrak{gl}(3, \mathbb{R})$, is given by the matrices E_{ij}, $i, j = 1, 2, 3$ defined by $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. In this basis we have

1. for $h_1 = h_2 = 0$, $h_3 \neq 0$,

   for $c_{11} = c_{22}$, $\mathcal{A}_p = \mathfrak{gl}(3, \mathbb{R})$ and $\mathcal{A}_{cp} = span\{E_{21} - E_{12}, E_{11} + E_{22} + 2E_{33}\}$,
   for $c_{11} \neq c_{22}$, $\mathcal{A}_p = \mathfrak{gl}(3, \mathbb{R})$ and $\mathcal{A}_{cp} = span\{E_{12}, E_{21}, E_{11} - E_{22}, E_{22} + E_{33}\}$;

2. for $h_2 = h_3 = 0$, $h_1 \neq 0$,

   $\mathcal{A}_p = \mathcal{A}_{cp} = span\{E_{23}, E_{32}, E_{22} + E_{33}, 2c_{11}E_{22} + c_{22}(E_{11} + E_{22})\}$.

Therefore the system is accessible for positive maps but not for completely positive ones in case 1 and it is never accessible in case 2. Thus different accessibility properties under the requests of positivity and complete positivity are possible, moreover Theorem 1 is not a sufficient condition for them.

4 Accessibility for a spin in a stochastic magnetic field

Consider a spin evolving under the action of a stochastic magnetic field $\vec{B}(t) = \langle \vec{B}(t) \rangle + \vec{\beta}(t)$, where $\langle \vec{B}(t) \rangle = (0, 0, B_3)$ is the time-independent average and $\vec{\beta}(t) = (\beta_1(t), 0, \beta_3(t))$ is a two-component stochastic part, with $\langle \vec{\beta}(t) \rangle = (0, 0, 0)$. This configuration can be obtained via a perfect shielding of the y component of the magnetic field. The control consists in switching on/off B_3 while the stochastic part is not affected by it. The two-time correlations of the stochastic components are given by

$$W_{ij}(t) = \langle \beta_i(t)\beta_j(0) \rangle, \tag{9}$$

entries of the real, positive definite covariance matrix $W(t)$. We assume that both diagonal and off-diagonal correlations functions are non-vanishing.

We can describe the time evolution of the spin system introducing a stochastic 2×2 density matrix ρ_s satisfying the semi-classical Liouville-Von Neumann equation

$$\dot{\rho}_s(t) = -i[B_3\sigma_3 + \vec{\beta}(t) \cdot \vec{\sigma}, \rho_s(t)] \tag{10}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Using the so-called convolutionless approach [13] it is possible to obtain the time evolution for the density matrix averaged over the stochastic component, $\rho(t) = \langle \rho_s(t) \rangle$:

$$\dot{\rho}(t) = -i[B_3\sigma_3, \rho(t)] - \sum_{k,l=1}^{3} \hat{c}_{kl}(t)[\sigma_k, [\sigma_l, \rho(t)]] \tag{11}$$

where

$$\hat{c}_{kl}(t) = \sum_{j=1}^{3} \int_0^t W_{kj}(s) U_{jl}(-s) ds \tag{12}$$

and the U_{jl} are the matrix elements of

$$U(t) = \begin{pmatrix} \cos 2B_3 t & -\sin 2B_3 t & 0 \\ \sin 2B_3 t & \cos 2B_3 t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

A Markovian approximation is justified whenever the coupling between the spin system and the external stochastic field is weak. It corresponds to neglecting the memory effects in (10), in practice $t \to +\infty$ in (12) and we obtain a time-independent Lindblad generator. The coefficients $c_{kl} = \hat{c}_{kl} + \hat{c}_{lk}$ define the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & 0 & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \tag{14}$$

where

$$\begin{aligned} c_{11} &= 2\int_0^{+\infty} W_{11}(s) \cos(2B_3 s) ds \\ c_{12} &= \int_0^{+\infty} W_{11}(s) \sin(2B_3 s) ds \\ c_{13} &= \int_0^{+\infty} W_{13}(s)(\cos(2B_3 s) + 1) ds \\ c_{23} &= \int_0^{+\infty} W_{13}(s) \sin(2B_3 s) ds \\ c_{33} &= 2\int_0^{+\infty} W_{33}(s) ds. \end{aligned} \tag{15}$$

In the coherence vector representation (11) becomes $\dot{\vec{\rho}} = -(\mathcal{H} + \mathcal{D})\vec{\rho}$ with

$$\mathcal{H} = 2\begin{pmatrix} 0 & B_3+\omega_3 & \omega_2 \\ -B_3-\omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{pmatrix}, \qquad \mathcal{D} = 2\begin{pmatrix} c_{33} & -c_{12} & -c_{13} \\ -c_{12} & c_{11}+c_{33} & -c_{23} \\ -c_{13} & -c_{23} & c_{11} \end{pmatrix}, \tag{16}$$

where

$$\begin{aligned} \omega_1 &= c_{23} \\ \omega_2 &= \int_0^{+\infty} W_{13}(s)(\cos{(2B_3 s)} - 1)ds \qquad (17) \\ \omega_3 &= -c_{12} \end{aligned} \tag{18}$$

The conditions of Theorem 1 are fulfilled since $K = span\{\vec{w} = (0,1,0)\}$, $dim\, K = 1$ and $C\vec{w} \neq 0$. The constraint of positivity does not affect the dimension of the space of parameters associated to this system whereas complete positivity asks for $c_{12} = c_{23} = 0$. In the latter case, the stochastic model is consistent only if the correlation functions are either vanishing, or white noise. All the possible cases are listed in the following table, together with the corresponding Lie algebras.

W_{11}(t)	$W_{13}(t)$	conditions	$\mathcal{A}_{cp}$
0	0	$c_{11} = c_{12} = c_{13} = c_{23} = 0$ $\omega_1 = \omega_2 = \omega_3 = 0$	$span\{E_{11} + E_{22}, E_{12} - E_{21}\}$
$W_{11}\delta(t)$	0	$c_{12} = c_{13} = c_{23} = 0$ $\omega_1 = \omega_2 = \omega_3 = 0$	$span\{E_{11}, E_{12}, E_{21}, E_{22}\} \oplus span\{E_{33}\}$
$W_{11}\delta(t)$	$W_{13}\delta(t)$	$c_{12} = c_{23} = 0$ $\omega_1 = \omega_3 = 0$	$\mathfrak{gl}(3, \mathbb{R})$

If positivity is imposed, no assumptions have to be done on the two-time correlation functions of the stochastic magnetic field and the Lie algebra is $\mathcal{A}_p = \mathfrak{gl}(3, \mathbb{R})$. Hence, the choice between positivity and complete positivity may affect the accessibility property of the system, depending on the assumptions on the correlations functions. The system is always accessible under the request of positivity, whereas it is accessible for a completely positive dynamics only if the stochastic magnetic field is assumed to have white noise correlations.

Conclusions

We discussed the impact of positivity and complete positivity of the dynamics on controllability and accessibility of a two-dimensional open system, evolving under a Markovian, entropy increasing time evolution. Whereas controllability is insensitive of what property is enforced, accessibility does. We gave in Theorem 1 a necessary condition for different accessibility and we discussed a concrete example in Section 4.

We stress that we considered a phenomenological approach to the dynamics of the system: the details of the interaction are unknown and a model of time evolution is assumed. Positivity or complete positivity are imposed after the Markovian approximation, leading

to some relations between the dissipative parameters that describe the dynamics and eventually constraining the microscopic properties of the surrounding. For example, in the case considered in Section 4 we observed that complete positivity is not compatible with generic two-time correlation functions of the bath.

In general, there can be transitions allowed if positivity is assumed and forbidden under complete positivity, since complete positivity implies a hierarchy in the relaxation times of diagonal and off-diagonal elements of the density matrix describing the system, that in turn affects the reachable sets of the system. However a different accessibility is a strongest condition, since it implies that the dimensions (as manifolds) of the reachable sets are different in the two cases.

Different accessibility properties have observable consequences. The measurement outcomes of some selected physical quantities can exhibit a dependence on whether positivity or complete positivity is asked for. A simple example is the following. Consider the spin in the stochastic magnetic field discussed in Section 4. Assume the initial state is polarized along the positive x direction, that is $\rho(0) = |\uparrow_x\rangle\langle\uparrow_x|$ or $\vec{\rho}(0) = (1/2, 0, 0)$. This state does not exhibit any polarization in spin along the z direction, S_z. By means of the switching on/off control u we want to get a (even slightly) polarized state along the positive z direction, that is a state for which the average of S_z does not vanish,

$$\langle S_z(t)\rangle = \mathrm{Tr}\Big(\frac{1}{2}\sigma_z\rho(t)\Big) = \rho_3(t) \neq 0. \tag{19}$$

It is convenient to evaluate

$$\frac{d}{dt}\langle S_z(t)\rangle = \dot{\rho}_3(t) = (\mathcal{L}\vec{\rho}(t))_3 \tag{20}$$

where $\mathcal{L} = -(\mathcal{H} + \mathcal{D})$. An explicit computation using the results of the previous section shows that, unless white noise correlations are assumed, for completely positive maps $\frac{d}{dt}\langle S_z(0)\rangle = 0$ whereas for positive maps $\frac{d}{dt}\langle S_z(0)\rangle \geqslant 0$. Moreover, since

$$\frac{d^n}{dt^n}\langle S_z(t)\rangle = (\mathcal{L}^n\rho(t))_3 \tag{21}$$

and considering that $\langle S_z(0)\rangle = 0$, it follows $\langle S_z(t)\rangle = 0$ for all time t for completely positive maps, whereas $\langle S_z(t)\rangle > 0$ at some t for positive maps. This behavior reflects the different accessibility properties in the two cases.

A final remark concerns possible generalizations of this work. We considered a Markovian approximation to the open system dynamics, and we assumed an entropy-increasing time evolution. If we try to relax these hypotheses, the characterization of controllability and accessibility is very difficult. For Markovian, entropy-increasing time evolutions these properties have been characterized in [21] under the request of complete positivity. It is not trivial to generalize this result to simple positive maps since for this class of maps there is not a satisfactory necessary and sufficient condition expressing positivity, equivalent to Remark 1.4. More generally, it would be very difficult to discuss the controllability and accessibility properties of the exact (non Markovian) open system dynamics, since it is described by a generalized master equation expressed by an integro-differential equation.

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